II

EVALUATION OF THE COEFFICIENTS

We shall extend to Dirichlet series the well known Cauchy formula for the coefficients of a Taylor series. We shall suppose that $\sigma_A < \infty$, although this condition is not essential. But we shall not have to use the most general cases in which the formulas, established here with this restrictive hypothesis, are still true.

THEOREM IX. *If*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

with $\sigma_A{}^f < \infty$, *then*

$$(15) \qquad a_n e^{-\lambda_n \sigma_1} = \lim_{T=\infty} \frac{1}{T}\int_{t_0}^{T} f(\sigma_1+it)e^{\lambda_n it}dt \quad (n \geqq 1),$$

where t_0 *is arbitrary, where* $\sigma_1 > \sigma_A{}^f$, *otherwise arbitrary, and where* $f(\sigma_1+it)$ *is the value of the principal branch of the function.*

We have with $s=\sigma_1+it$:

$$\frac{1}{T}\int_{t_0}^{T} f(s)e^{\lambda_n s}dt = \frac{1}{T}\int_{t_0}^{T}\left(\sum_{1}^{\infty} a_m e^{(\lambda_n-\lambda_m)s}\right)dt$$

$$=\frac{1}{T}\int_{t_0}^{T}\left(\sum_{1}^{n-1} a_m e^{(\lambda_n-\lambda_m)s}\right)dt + \frac{1}{T}\int_{t_0}^{T} a_n dt + \frac{1}{T}\int_{t_0}^{T}\left(\sum_{n+1}^{\infty} a_m e^{(\lambda_n-\lambda_m)s}\right)dt.$$

Since, for k real, $k \neq 0$:

$$\lim_{T=\infty} \frac{1}{T}\int_{t_0}^{T} e^{k(\sigma_1+it)}dt = 0,$$

and since the series under the integral signs converge uniformly with respect to $t(-\infty < t < \infty)$, we see that:

$$\lim_{T=\infty} \frac{1}{T}\int_{t_0}^{T}\left(\sum_{1}^{n-1} a_m e^{(\lambda_n-\lambda_m)s}\right)dt = 0$$

$$\lim_{T=\infty} \int_{t_0}^{T} \left(\sum_{n+1}^{\infty} a_m e^{(\lambda_n - \lambda_m)s} \right) dt = 0.$$

Hence

$$\lim_{T=\infty} \frac{1}{T} \int_{t_0}^{T} f(s) e^{\lambda_n s} dt = \lim_{T=\infty} \frac{1}{T} \int_{t_0}^{T} a_n dt = a_n,$$

which is equivalent to the statement of the theorem.

As a matter of fact, it is seen that in Theorem IX, $\sigma_A{}^f$ can be replaced by $\sigma_u{}^f$, since only the uniform convergence was used in the proof.

We shall have to use the following lemma.

LEMMA I. *If $c>0$, if k is a positive integer, and if ω is real, then:*

$$(16) \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(c+it)}}{(c+it)^k} dt \left\{ \begin{array}{ll} = \dfrac{\omega^{k-1}}{(k-1)!}, & (\omega > 0) \\ = \quad 0, & (\omega \leqq 0) \end{array} \right. .$$

Let us first suppose $\omega > 0$. If $T>0$, $\sigma_1 < 0$, we shall denote by $I_1\ (\pm T,\ \sigma_1)$ the segments $(\sigma_1 \leqq \sigma \leqq c,\ t = \pm T)$ and by $I_2(T,\sigma_1)$, $I_3(T)$ the segments $(\sigma = \sigma_1,\ |t| \leqq T)$ and $(\sigma = c,\ |t| \leqq T)$. By $C(T,\sigma_1)$ we shall denote the rectangle composed of the four segments. We have by the theorem on residues:

$$(17) \qquad \frac{1}{2\pi i} \oint_{C(T,\sigma_1)} \frac{e^{\omega s}}{s^k} ds = \frac{\omega^{k-1}}{(k-1)!}.$$

Since for σ_1 fixed the integrals extended over $I_1(T,\sigma_1)$ and $I_1(-T,\sigma_1)$ tend to zero, we see that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(c+it)}}{(c+it)^k} dt - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(\sigma_1+it)}}{(\sigma_1+it)^k} dt = \frac{\omega^{k-1}}{(k-1)!},$$

and since the second integral in this equality tends to zero as σ_1 tends to $-\infty$, we see that our equality holds for $\omega > 0$. The proof is similar if $\omega \leqq 0$. Here $C(T,\sigma_1)$ should be replaced by a contour $C'(T,\sigma_2)$ with $\sigma_2 > c$ and composed of segments $I_1'\ (\pm T,\sigma_2) \equiv (c \leqq \sigma \leqq \sigma_2,\ t = \pm T)$, $I_2(T,\sigma_2) \equiv (\sigma = \sigma_2,$

$|t| \leqq T)$, $I_3(T)$. The integral (17), in which $C(T,\sigma_1)$ is replaced by $C'(T,\sigma_2)$, is then zero, since the function under the integral sign is holomorphic inside $C'(T,\sigma_2)$, and on making first T tend to $+\infty$ and then σ_2 to $+\infty$ we obtain the desired formula.

THEOREM X. *If*

$$f(s) = \sum a_n e^{-\lambda_n s}$$

with $\sigma_A{}^f < \infty$, *if* $\nu \geqq 0$ *and if* k *is a positive integer, then*

$$\left.\begin{array}{ll} \text{for } \nu > \lambda_1: & \sum\limits_{\lambda_n < \nu} (\nu - \lambda_n)^{k-1} a_n \\ \text{for } 0 \leqq \nu \leqq \lambda_1: & 0 \end{array}\right\} = \frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \frac{f(c+it)e^{\nu(c+it)}}{(c+it)^k} dt$$

where $c > \max(\sigma_A{}^f, 0)$.

We have, with $s = c + it$ for $\nu > \lambda_1$:

$$\int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \int_{-\infty}^{\infty} \frac{(\sum a_n e^{-\lambda_n s}) e^{\nu s} dt}{s^k}$$

$$= \int_{-\infty}^{\infty} \frac{\sum\limits_{\lambda_n < \nu} a_n e^{(\nu - \lambda_n)s}}{s^k} dt + \int_{-\infty}^{\infty} \frac{\sum\limits_{\lambda_n \geqq \nu} a_n e^{(\nu - \lambda_n)s}}{s^k} dt.$$

From Lemma I and from the uniform convergence of the series under the last integral sign we see that the last integral is zero. Therefore by Lemma I:

$$\frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \frac{f(s)}{s^k} e^{\nu s} dt = \frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{\lambda_n < \nu} a_n e^{(\nu - \lambda_n)s} \right) \frac{dt}{s^k}$$

$$= \frac{(k-1)!}{2\pi} \sum_{\lambda_n < \nu} a_n \int_{-\infty}^{\infty} \frac{e^{(\nu - \lambda_n)s}}{s^k} dt = \sum_{\lambda_n < \nu} a_n (\nu - \lambda_n)^{k-1}.$$

If $\nu \leqq \lambda_1$ then

$$\int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \int_{-\infty}^{\infty} \left(\sum_{\lambda_1 \geqq \nu} a_n e^{(\nu - \lambda_n)s} \right) \frac{dt}{s^k} = 0.$$

In this theorem, too, $\sigma_A{}^f$ can be replaced by $\sigma_u{}^f$.